The Banjo: the “Model” Instrument

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If you stare at a banjo hard enough, you can see two interacting, wave-bearing systems; specifically, strings and a circular membrane. What’s more, they are systems for which we can solve the equations of motion and thereby describe the propagation of structural vibrations; i.e. waves. Why this hasn’t been done before is a mystery to me.

At its lowest level of abstraction, the “simplified” banjo shown above is basically two interacting vibrating systems: a plucked string and a circular membrane. By “interacting” I mean that waves in the string generate waves in the membrane, and visa versa, so that the whole instrument vibrates together. The five-string banjo is a little more complicated as it has six interacting systems -- five strings and one membrane. The equations which describe the way waves travel in these systems, how they radiate sound, and how they interact are fairly straightforward. For example, a string has only two coordinates, position and time, and the equation which describes the propagation of a disturbance along the string, the *wave equation*, links these two variables in such a way that if you know the wave’s position at any time, you know it at *any* time. It’s an analogous situation in the membrane but complicated somewhat by the bridge being placed away from the the membrane’s geometric center.

Another complication, common to both the strings and the membrane, is that every a wave hits something, it changes as it reflects and/or passes through. This complicates things considerably and the math can get pretty messy.[1] Part of the “art” of modeling is to decide where you can simplify things without throwing out the essence of what you’re looking for. For example, there are undoubtedly waves traveling in the neck as though it were a sixth string: are they important? And there are also reverberations within the bridge, tone ring, and other parts: are they important? As banjo modeling gets more sophisticated, there are many such considerations which will become important, and with this will come more pressure to simplify while retaining what you set out to discover. In this sense, developing a model is a little like collecting garbage, you really should know what you are going to do with it before you start. The model which I used for the results below, approximated the influence of the clamp and tone ring, bridge, nut, etc. on the waves in the string and head but did not consider the reverberation of waves within them.

Why do we go to so much trouble to model the banjo? Partly to build better banjos or help people who own and play banjos get a better sound by understanding how the various components of the instrument work together and influence the sound. More generally, we do this sort of thing [modeling] to build a better *anything*. This story describes the why and wherefore of analytical modeling.

The strings. Given the tension in the string (provided by the tuning peg) and its mass (diameter) we know how an initial displacement (a *pluck*) travels and reflects off the ends. We call this a structural vibration, or a wave, and this wave will travel back and forth in the string reflecting alternately from one end and the other. And just as it takes some force to contain the end of a jumping rope, it takes force to keep the string connected (through the bridge) to the membrane. We can calculate all of this. Figure 2 describes a pluck in the upper part of the figure as, initially, a triangular shaped displacement shown as a dotted line. The vertical scale in the figure is greatly exaggerated to make it easier to see what is going on. Normally, the displacement of the string is only about one-thousandth of its length. The figure also shows what the string shape would be after about 12 reverberations. Note that the response of the string has diminished in time because there has been energy shared with the other systems and because all the systems have losses. Note, also, that the triangular shape of the pluck has smoothed to the point of looking like our jumping rope; specifically, it resembles the sinusoidal shape we generally expect from a vibrating string. There are several factors which contribute to this smoothing effect. The largest is that the strings are not completely flexible; they have some stiffness that causes the various frequency components in the wave to travel at

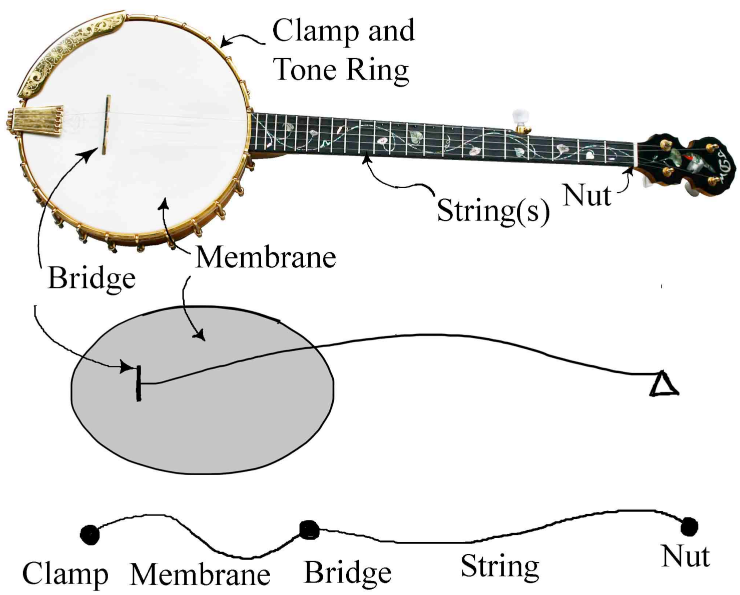

Figure 1. The banjo simplified. Breaking the banjo down into interacting parts that are individually solvable.

different speeds.[2] This disperses or spreads the wave shape as it propagates and is called "dispersion"!

Back to the jumping rope analogy. You may recall that it takes some force on your part to hold the end of the rope. For a plucked string, the force analogous to your hand is supplied by the membrane. I have calculated this force and show it in the lower part of the Figure. This force, when transferred through the bridge to the membrane, causes the membrane to deflect and radiate sound. The force varies with time, and this variation depends on the shape of the wave hitting the bridge, and this, in turn, is determined by where the string is plucked. The figure shows two illustrative cases: in the top case the string is plucked as shown in the top part of the Figure and in the other case the string is plucked mid-way between the bridge and the nut. Plucked at the center, the initial shape of the string is a little more like a (half) sine wave than the top curve; and so the imparted force and the shape settle in to a sine-wave shape more quickly, i.e. the string settles into vibrating at a specific frequency that we hear as a particular note. We call this a *mode.* The settling-in of the pluck to this mode is more apparent in the slow motion movie version of the evolution of the pluck in time. [Movie 1-StringResp] Finally, the settling-in process takes only a small fraction of a second and during that time, the vibration is not modal and, therefore, not a well defined note. This is the "twang".

The membrane. Like the string, the membrane is also under tension and has mass but in this case waves travel as circles expanding outwardly from their origin rather than along a line. This would be orderly enough except that their origin (the bridge) is not in the center of the circle. As a result, different parts of the expanding wave meet the boundary (the clamp) at different times. It's much easier to visualize this added complexity by looking at the propagation of a very short wave. Such a wave would be impossible to generate experimentally, but in the world of mathematics, we can do pretty much anything. Thus, Figure 3 shows a snapshot of such a calculated wave which has left the source and traveled to a point where part of the wavefront has reflected from the clamp. Note that the vertical scale is exaggerated and that the pulse goes from positive amplitude (i.e. upward) to negative (downward) at the reflection. Every reflection at the clamped edge will cause such a reversal. Also, keep in mind that the wave looses energy (and amplitude) as it travels and when it encounters either the clamp or the bridge. Again, the complexity is better appreciated in the time evolution movie: [Movie 2-ShortPulse].

The ability to generate such a non-physical pulse and to readily change material properties illustrates several of the major benefits of modeling. In this case it illustrates the complexity of a pulse reverberating in a circular membrane. In other cases, it allows one to play the "what if" game by adjusting parameters with just a few keystrokes. A word of caution here; the "what if"

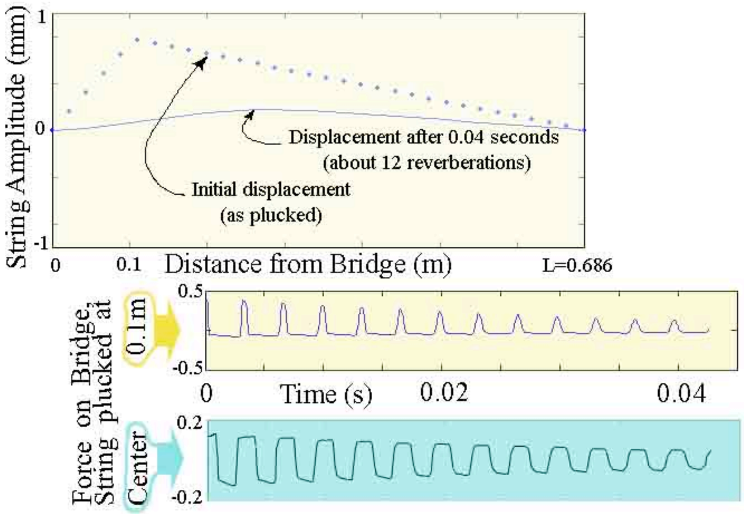

Figure 2. The calculated amplitude of the string along its length, when plucked, and after about twelve reverberations showing the string "settling in" to a modal shape -- a tone.

game can be very irksom to experimentalists who are constrained to real-world materials. Admittedly, the many approximations which go into this and most modeling, mean that mathematical models can never be exact. The process is, however, often useful to predict if a proposed change is leading you in the intended direction.

Replacing the short pulse with a more realistic excitation imparted to the head by the string, we get head displacements which look quite different and are, in fact, similar to measured results. A snapshot of a calculated head displacement while the head is being driven by a damped sinusoidal oscillation is shown in Figure 4. Here the frequency of the drive is close to one of the natural resonance frequencies of the head. The movie of the head response [Movie 3-LongPulse] covers only a fraction of a second in actual time. During this time you can observe the head excitation build for a few tens of milliseconds, and then slowly diminish as the drive diminishes and the head dissipates energy. Calculations also allow one to compare the energy lost to material damping and sound radiation.

The head tension. Another advantage of studying the banjo, is that many of its set-up parameters are easily adjusted and many of its parts are easily replaced. This is one reason that so many banjo players fiddle with their banjo! One of the most influential parameters is the head tension. Although I don't do much experimental work any more, I do appreciate the need for it and particularly the need for "ground truth" when modeling. So I took my banjo, tapped the head with a screwdriver, and recorded the sound. This "delta function" excitation should, in principle, excite all the head modes and a spectral analysis of the recorded sound should display these modes. The results are shown in Figure 5.

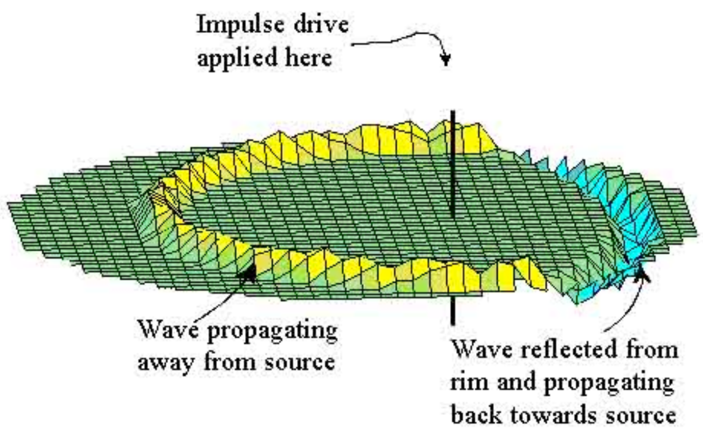

Figure 3. A "snapshot" of a hypothetical short pulse on a banjo head. The pulse originated at the bridge (the vertical line) and has traveled as an expanding circle, except for that part of the wavefront which has encountered and reflected away from the back edge of the membrane.

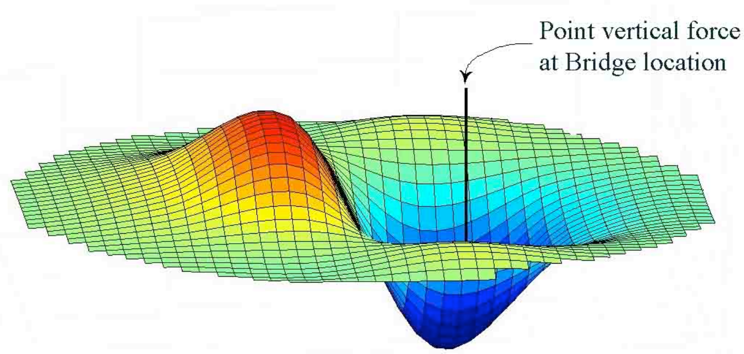

Figure 4. The highly exaggerated deflection of the head when driven at the bridge position by a damped sinusoidal force. The frequency of the sinusoid is near a resonance of the head.

To calculate the modal response for comparison, I calculated the average magnitude of the head response for harmonic (single frequency) drives as a function of frequency. This would not be exactly the same quantity measured but should show show the same frequency peaks. The comparison of measured peaks and calculated ones are shown in Figure 5.

I could not find tabulated values for the flexural wave speed in mylar membranes under various tensions, so I adjusted the tension in the calculation (a single number in the computer code) until the first peak at 295 Hz. matched the measured one; voila, all the other peaks lined up! Also, the speed which did the matching, 150 m/s, is exactly what was used by Rae and Rossing [3] in their measurements of vibration mode patterns of banjo heads. [4]

There are several aspects of Figure 5 which are of interest. One is that the peaks are not harmonic in the sense that all the higher resonances are multiples of the first harmonic at 295 Hz. Also, the double peak near the third resonance is in both the measured and calculated spectra. I don't know what is causing this. Lastly, there is an exponential "background" in the calculated data which adds a sort of floor to the data and I don't know where that comes from either. If this were subtracted from the response the agreement between experiment and calculation would be remarkable.

Finally, the sound. A significant, and certainly the most pertinent energy loss to the membrane waves is the energy radiated as sound. As a section of the membrane moves up, it pushes and condenses the air above it causing a slight increase in ambient pressure; as it subsequently moves down, it forms a rarefaction, a slight decrease from ambient pressure. This alternating pressure/rarefaction is, by definition, a sound wave. The perceived loudness of this sound wave, or the signal recorded by a microphone at some distance from the banjo, is proportional to the average response of the membrane surface.

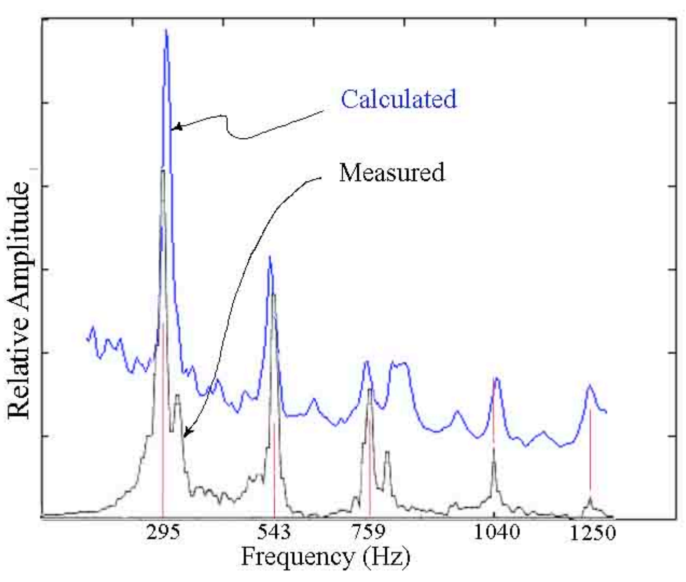

Figure 5. A comparison of the measured and calculated head frequency response.

Here's the procedure for calculating the radiated sound from a plucked banjo:
1) calculate the force (as a function of time) exerted on the banjo head by the plucked string,
2) use this force to excite waves in the head,
3) for each time interval, average (integrate) the response over the surface of the head,
4) plot this as a function of time.

This brings us to Figure 6 which uses the above prescription with the 1st string plucked at 0.1 m from the bridge. This is the bottom string in the banjo pictured in Figure 1 and is tuned to 293 Hz.

(sometimes referred to as D_4).[5] The tension in the calculation was adjusted so that the first head resonance agreed with the measured one at 295 Hz.

There are several interesting aspects of Figure 6. First, note that radiated sound increases for the first 30 to 40 ms while energy is being transferred from the string to the membrane. Also note that there are two distinct decay curves: one lasting about 200 ms and the other still persisting at 1 s. It is also interesting that the waveform is much “cleaner” (i.e. sinusoidal) after the instrument settles into the slow decay mode. It is conjecture on my part, but I suspect that the “ragged” waveform in the early stages of sound radiation accounts for the “twang” associated with the classic banjo sound, and the nearly tonal waveform later is the ringing.

So there you have it, summarized in a sentence. A string held at both ends is plucked; it interacts with and causes waves in a membrane, which radiates sound. It may seem like a lot of trouble for something which can be summarized in one sentence; but here’s the deal. Once the model is on the computer, it takes only a few seconds to calculate allowing you to easily change things like head tension, string gauge, bridge mass, etc. to see what happens to the sound.

And now the bad news. Real world systems are generally too complicated to be modeled accurately. There are a number of “simplifying assumptions” in this and virtually all analytical models. Also, in the case of musical instruments, analytical models compete with the most sophisticated acoustic analyzer of all: the ear. Models and experiments are a long way from discerning the subtleties of “good tone”. The evolution of good tone in musical instruments has been largely trial and error and, as such, has been a slow process. Analytical modeling offers a useful tool to estimate the sensitivity of tonal parameters to proposed design changes and possibly shorten the design cycle. There is “sound at the end of the tunnel”.

References and endnotes:

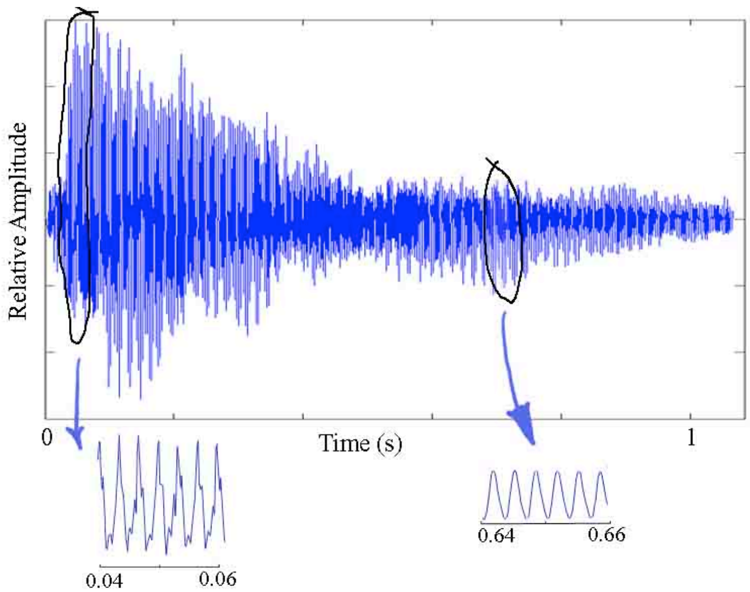

Figure 6. The calculated sound radiation from the banjo. The detail excerpts show the evolution of the sound from a waveform with significant harmonic content to a nearly sinusoidal shape; i.e. from "twang" to "ring".

1.Joe Dickey, "The structural dynamics of the American five-string banjo", Journal of the Acoustical Society of America **114**, 2958-2966 (2003).

2.John Bryner, Stiff-string theory: Richard Feynman on piano tuning, Physics Today, December 2009. This is a cute article describing a letter from Richard Feynman to his piano tuner in which he expounds on the stiffness of strings and their influence on tuning.

See also: Neville Fletcher and Thomas Rossing,*The Physics of Musical Instruments*, 2nd Edition (Springer 1998), Section 10.4.3.

3.J.Rae and T. Rossing, Design aspects of the five string banjo (A), Journal of the Acoustical Society of America, **117**, 2590 (2005).

4. Thomas R. Moore and Laurie A. Stephey, Time-resolved studies of banjo head motion (A), Journal of the Acoustal Society of America, **127**, No. 3, Pt. 2, 1870 (2010).

5. It is not considered good practice to have the head resonance this close to a string resonance, as this seems to produce a very loud note -- something like the dreaded "wolf tone" sometimes seen in other stringed instruments. Of course, with the banjo, getting rid of this is just a matter of adjusting the head tension with the tensioning nuts seen around the clamp ring in Figure 1. Presto, you have moved the head resonance to wherever you want it. Don't try this with your cello.

My friends laughed at me when I told them that I wanted to go into modeling as a career. Well here I am, only about 35 years out of grad. school and still holding true to my mantra: "If it has no possible use, it must be good science." I did have a career doing useful work helping make submarines quiet for the Navy, and then a decade with Johns Hopkins University doing research. Altogether I've published about a hundred papers, mostly in structural vibrations. About a third of these were co-authored with the late Gideon Madanik. I spent an interesting year in 1983-84 as Congressional Science Fellow, was chair of the ASA Membership Committee for almost a decade, represented the ASA and AIP in various international committees. I'm now pretty much retired but still writing a few papers, tending an American Chestnut Foundation orchard on our farm in Davidsonville, Maryland, doing and teaching woodturning, and, oh yes, I play banjo in a couple of local bluegrass bands.